Tutorial:

2.1.1.

2.2.1

2.2.11

2.1.27

Problem set:

2.1.4 2.2.2

2.1.5 2.2.7

2.1.7 2.2.8

2.1.9 2.2.11

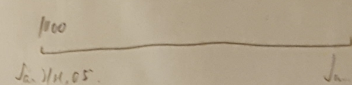
2.1.12

2.1.33

2.1.4.

$1000\ddot{s}_{\overline{300}|.01} = Y\ddot{a}_{\overline{300}|0.01} \Rightarrow Y = \dfrac{1000\ddot{s}_{\overline{300}|0.01}}{\ddot{a}_{\overline{300}|0.01}} = 1000 \cdot \dfrac{\frac{(1+i)^{300}-1}{d}}{\frac{1-\left(\frac{1}{1+i}\right)^{300}}{d}} = 1000(1+i)^{300}$.

$i = \frac{i^{(12)}}{12} = 0.01$, $Y = 19{,}788.47$.



2.1.5.

i) $100\, s_{\overline{7}|0.0075} = 715.95$, $\frac{i^{(12)}=9\%}{12} = 0.0075$

ii) $100\, s_{\overline{19}|0.0075} = 2033.87$, $\frac{i^{(12)}=9\%}{12} = 0.0075$.

iii) $i_1 = \frac{0.105}{12} = 0.00875$, $i_2 = \frac{0.12}{12} = 0.01$,

$100\left[s_{\overline{19}|0.00875} \cdot (1.00875)^{4}(1.01)^{4} + s_{\overline{4}|0.00875}(1.01)^{4} + s_{\overline{4}|0.01}\right] = 3665.12$.

iv) $3665.12 \times 0.01 = 36.65$.

2.1.7.

a) $10(1.05)^{30} \cdot s_{\overline{10}|.05} + 20(1.05)^{20} s_{\overline{10}|.05} + 30(1.05)^{10} s_{\overline{10}|.05} + 40\, s_{\overline{10}|.05} = 2328.82$

b) ~~$(s_{\overline{40}|} + s_{\overline{30}|.05} + s_{\overline{20}|.05} + s_{\overline{10}|.05})$~~ =

$10\left[s_{\overline{40}|} - s_{\overline{30}|} + 2(s_{\overline{30}|} - s_{\overline{20}|}) + 3(s_{\overline{20}|} - s_{\overline{10}|}) + 4 s_{\overline{10}|}\right] = 10(s_{\overline{40}|.05} + \cdots + s_{\overline{10}|.05})$.

$s_{\overline{40}|} - s_{\overline{30}|} = (1.05)^{30} \cdot s_{\overline{10}|} \Rightarrow \frac{(1+i)^{40}-1}{i} - \frac{(1+i)^{30}-1}{i} = (1+i)^{30} \cdot \frac{(1+i)^{10}-1}{i}$

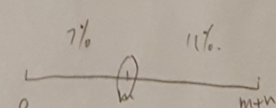
2.1.9.

$I_t = 1 \cdot i \cdot s_{\overline{t-1}|i} = i \cdot \frac{(1+i)^{t-1}-1}{i} = (1+i)^{t-1}-1$

$\sum_{t=1}^{n} I_t = \sum_{t=1}^{n}\left[(1+i)^{t-1}-1\right] = \sum_{t=1}^{n}(1+i)^{t-1} - n = \frac{(1+i)^{t}-1}{i} - n = s_{\overline{n}|i} - n$.

Total interest = total accumulated value − total deposit.

$1 \cdot s_{\overline{m}|7\%} \cdot (1+11\%)^{n}$



2.1.12.

$A - V = 1 \cdot s_{\overline{m}|.07} \cdot (1+11\%)^{n} + 1 \cdot s_{\overline{n}|.11} = 128 + 34 \cdot (1.11)^{n}$

$s_{\overline{n}|.11} = \frac{(1.11)^{n}-1}{0.11} = 128 \Rightarrow (1.11)^{n} = 15.08 \Rightarrow A - V = 690.72$.

2.1.33.

(a)

$v^{t} \cdot s_{\overline{n}|} = (1+i)^{-t} \cdot \frac{(1+i)^{n}-1}{i} = \frac{(1+i)^{n-t}-(1+i)^{-t}}{i},$

$a_{\overline{t}|} = \frac{1-v^{t}}{i} = \frac{1-(1+i)^{-t}}{i}, \quad s_{\overline{n-t}|} = \frac{(1+i)^{n-t}-1}{i},$

$\ddot{a}_{\overline{t}|} = (1+i)\,a_{\overline{t}|}, \quad \ddot{s}_{\overline{n-t}|} = (1+i)\,s_{\overline{n-t}|}$

$\ddot{a}_{\overline{t}|} + \ddot{s}_{\overline{n-t}|} = (1+i) \cdot \frac{1-(1+i)^{-t-1}}{i} + (1+i) \cdot \frac{(1+i)^{n-t+1}-1}{i}$

(b)

$(1+i)^{t}\, \ddot{a}_{\overline{n}|} = (1+i)^{t} \cdot (1+i)\, a_{\overline{n}|} = (1+i)^{t+1} \cdot \frac{1-(1+i)^{-n}}{i} = \frac{(1+i)^{t+1}-(1+i)^{t+1-n}}{i}$

$\ddot{s}_{\overline{t}|} + \ddot{a}_{\overline{n-t}|} = (1+i)\, s_{\overline{t}|} + (1+i)\, a_{\overline{n-t}|} = (1+i) \cdot \frac{(1+i)^{t}-1}{i} + (1+i) \cdot \frac{1-(1+i)^{t-n}}{i}$

$s_{\overline{t+1}|} + a_{\overline{n-t-1}|} = \frac{(i+1)^{t}-1}{i} + \frac{1-(1+i)^{-(n-t+1)}}{i}.$

2.2.2

Derek: $1200\, \ddot{s}_{\overline{20}|0.06} = 1200(1+0.06)\, s_{\overline{20}|0.06} = 1200 \times 1.06 \times \frac{(1.06)^{20}-1}{0.06} = 69{,}787.66$

Anne: $1200\, s_{\overline{20}|0.06} = \frac{69{,}787.66}{1.06} = 65{,}837.41$

Ira: $100\, \ddot{s}_{\overline{240}|j} \qquad j = (1.06)^{\frac{1}{12}} - 1. \qquad 100\, \ddot{s}_{\overline{240}|j} = 67{,}958.16$

2.2.7.

~~42000~~

$10{,}000(1.05)^{n} \cdot 0.05 \geq 2000 \implies n \geq 28.4.$

3800

At time 29 $\quad 10{,}000(1.05)^{29} = 41{,}161.36.$

$\frac{95}{1900}$

smaller scholarship $1{,}161.36.$

38000 + 1900

$(10{,}000(1.05)^{n} - 2000) \cdot 0.05 \geq 2000,$

2.2.8.

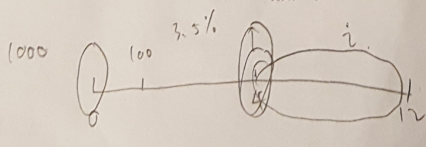
$i = (1.0075)^{12} - 1 = 0.09.$

AV: $100\, \ddot{s}_{\overline{12n}|j} + 1000\, s_{\overline{n}|i} \geq 100{,}000 \implies \frac{(1+i)^{n}-1}{d_j} + 10 \cdot \frac{(1+i)^{n}-1}{i} \geq 1000$

$\implies n \geq 18.3. \quad n = 19. \quad \ddot{s}_{\overline{228}|j} + 10\, s_{\overline{19}|i} = 1078.2.$ April 30th.

2.2.11.

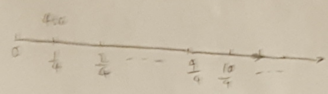
$1000 = 100\,[a_{\overline{4}|0.035} + v^{4}_{0.035}\, a_{\overline{8}|i}] \implies a_{\overline{8}|i} = 7.26. \implies i = 2.208\%.$



$100\, a_{\overline{8}|i} \cdot v^{4} + 100 \cdot a_{\overline{4}|3.5\%} = 1000$

Problem Set 4: 2.2.3. 2.2.4. 2.2.8. 2.2.10. 2.2.16. 2.2.13. 2.2.20. (2.2.5. 2.2.6. 2.2.9)

2.2.3.



annual rate is 7%, quarter rate is $(1+i)^4 = 1+7\%$, $\Rightarrow i = (1.07)^{\frac{1}{4}} - 1 = 0.017$

$450\, s_{\overline{40}|i} = Y \ddot{a}_{\overline{4}|7\%} \Rightarrow Y = 9872.$

$\uparrow$ end $\qquad\qquad$ $\uparrow$ beginning

2.2.4

4-year rate: $j = (1+i)^4 - 1$

$100\, \ddot{s}_{\overline{10}|(1+i)^4-1=j} = X$

$100\, \ddot{s}_{\overline{5}|(1+i)^4-1=j} = \frac{X}{5}$ $\Rightarrow j = 0.3195, \quad X = 6195.$

2.2.8.

Annual rate: $i = (1+\frac{i^{(12)}}{12})^{12} - 1 = 0.094$, monthly rate: $j = \frac{i^{(12)}}{12}$.

$100\, \ddot{s}_{\overline{12n}|j} + 1000\, s_{\overline{n}|i} \geq 100{,}000$ $\qquad$ n is number of month.

$\Rightarrow n \geq 18.3, \quad n = 19.$ $\qquad$ Apr 30th, 2013.

2.2.10.

Account A: $1000\, \ddot{s}_{\overline{10}|0.05} (1.05)^{n-10} \leq 500\, \ddot{s}_{\overline{n}|0.05} \Rightarrow n \geq 30.32.$

Jan 1st, 2015.

2.2.16.

11 month payments are 1, last month is 2, which is equivalent to

12 month of payment 1 plus annual payment 2.

monthly rate is j, annual rate is $i = (1+j)^{12} - 1 = j \cdot \frac{(1+j)^{12}-1}{j} = j\, s_{\overline{12}|j}.$

month present value: $\frac{1}{j}$. annual present value: $\frac{1}{i}$, $(\frac{1}{i} + \frac{1}{j}) = \frac{1}{j}(1 + \frac{1}{s_{\overline{12}|j}})$

2-2-13.

$$12{,}000 = 592.15\, a_{\overline{24}|j} = 426.64\, a_{\overline{36}|j}.$$

$$\Rightarrow j = 0.0140. \quad i^{(12)} = j \times 12 = 0.168.$$

$$12{,}000 = K \cdot a_{\overline{48}|j} \Rightarrow K = 345.02.$$

$$592.15 \cdot \frac{1-v_j^{24}}{j} = 426.64 \cdot \frac{1-v_j}{j}$$

$$\cancel{\frac{592.15}{426.64} = \frac{1-v_j^{36}}{1-v_j^{24}}}$$

$$592.15 - 592.15\, v_j^{24} = 426.64 - 4$$

$$592.15 - 426.64 = v_j^{24}(592.15$$

$$x = v_j^{12}, \quad x^2 = v_j^{24}$$

2-2-20.

(a) $1 \cdot s_{\overline{20}|3\%} \cdot (1+4\%)^n + 1 \cdot s_{\overline{n}|4\%} \geq 100. \Rightarrow n \geq 22.4. \Rightarrow n = 23.$

(b) $1 \cdot s_{\overline{n}|3\%}\,(1+4\%)^n + 1 \cdot s_{\overline{n}|4\%} \geq 100 \Rightarrow n = 22.$ (the equation is 106.618 when $n = 22$).

12%

$$\frac{(1+3\%)^n - 1}{3\%} \cdot (1+4\%)^n + \frac{(1+4\%)^n - 1}{4\%} \geq 100$$

$$[4(1+3\%)^n - 4] \cdot (1+4\%)^n + 3[(1+4\%)^n - 1] = 12$$

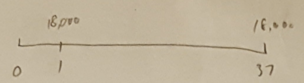
$$4(1+3\%)^n(1+4\%)^n - 4 \cdot (1+4\%)^n + 3(1+4\%)^n = 15$$

$x^n y^n = (xy)^n$

$$4 \cdot (1.03 \cdot 1.04)^n - 1.04^n = 15.$$

2.3.8

(a)



$18.000\,(1+(1+4\%)+\cdots+(1+4\%)^{36}) = 18{,}000\ s_{\overline{37}|4\%} = 1{,}470{,}640.$

average salary $= \dfrac{1{,}470{,}640}{37} = 39.747.$

Pension $= 0.70 \times 39.747 = 27{,}823.$

(b)

$0.75 \times 37 \times 39{,}747 = 36{,}766$

(c) $18{,}000\,[(1+4\%)^{36}+\cdots+(1+4\%)^{27}] \times \dfrac{1}{10} = 62{,}312.$

Pension $= (0.25) \times 37 \times 62.312 = 57{,}639.$

(d) annual ~~payment~~ deposit is $18{,}000\,(1+4\%)^{n} \cdot 3\%$

$AV = 18{,}000 \cdot 6\% \cdot (1+6\%)^{36} + 18{,}000 \cdot (1+4\%) \cdot 6\% \cdot (1+6\%)^{35} + \cdots + 18{,}000\,(1+4\%)^{36} \cdot 3\%$

$= 6\% \times 18{,}000 \times (1.06)^{36} \cdot \left(\dfrac{1-\left(\frac{1.04}{1.06}\right)^{37}}{1-\frac{1.04}{1.06}} \right) = 247{,}845.$

$247{,}845 = X \cdot \ddot{a}_{\overline{20}|6\%} \Rightarrow X = 19{,}974.$

2.3.11.

Sandy: $PV = 100v + (100+10)v^2 + \cdots + (100+10n)v^n + \cdots$

$= 90(v+v^2+\cdots) + 10v + 20v^2 + \cdots$

$= 90\,a_{\overline{\infty}|i} + 10 \cdot (Ia)_{\overline{\infty}|i} = 90 \cdot \frac{1}{i} + 10\left(\frac{1}{i}+\frac{1}{i^2}\right)$

Danny: $PV = 180\,\ddot{a}_{\overline{\infty}|i} = \dfrac{180}{d} = \dfrac{180(1+i)}{i}$; $\dfrac{180(1+i)}{i} = \dfrac{90}{i} + 10\left(\frac{1}{i}+\frac{1}{i^2}\right) \Rightarrow i = 10.2\%.$

2.3.12.

$X = 2(Ia)_{\overline{60}|j}$, j is monthly rate, 3-month rate is $\frac{9\%}{4} = 2.25\%.$

$(1+j)^3 = 1+2.25\% \Rightarrow j = 0.007444$

$X = 2 \cdot \dfrac{\ddot{a}_{\overline{60}|j} - 60v_j^{60}}{j} = 2729.$

Tutorial: 3.1.1, 3.1.4, 3.2.5; Problem Set 3.1.2, 3.1.6, 3.1.9, 3.1.10, 3.2.1, 3.2.2, 3.2.3, 3.2.4

3.1.2.

$OB_t = OB_0(1+i)^t - K_1(1+i)^{t-1} - K_2(1+i)^{t-2} - \cdots - K_{t-1}(1+i) - K_t$

$= (K_1 v + K_2 v^2 + \cdots + K_n v^n)(1+i)^t - K_1(1+i)^{t-1} - \cdots - K_t$

$= K_{t+1} v + K_{t+2} v^2 + \cdots + K_n v^{n-t}$

monthly rate $i = \frac{9\%}{12} = 0.75\%$, $v_{0.0075} = \frac{1}{1+0.75\%}$

Third 20 payments $K_{41} = 1000(1-2\%)^{40}$, $K_{42} = 1000(1-2\%)^{41}$, $\cdots$, $K_{60} = 1000(1-2\%)^{59}$.

$OB_{40} = 1000 \cdot 0.98^{40} \cdot v_{0.0075} + 1000 \cdot 0.98^{41} \cdot v^2_{0.0075} + \cdots + 1000 \cdot 0.98^{59} \cdot v_{0.0075}$

$= 1000 \cdot 0.98^{40} v_{0.0075} [1 + 0.98 v_{0.0075} + 0.98^2 v^2_{0.0075} + \cdots + 0.98^{19} v^{19}_{0.0075}]$

$= 1000 \cdot 0.98^{40} \cdot v_{0.0075} \cdot \frac{1-(0.98 v_{0.0075})^{20}}{1-0.98 v_{0.0075}} = 6889.$

3.1.6.

$L = OB_0 = (OB_0 - OB_1) + (OB_1 - OB_2) + \cdots + (OB_{n-1} - OB_n)$

$= PR_1 + PR_2 + \cdots + PR_n$

$= (K_1 - I_1) + (K_2 - I_2) + \cdots + (K_n - I_n) = K_T - I_T.$

$OB_0 = L$

outstanding balance $OB_{t+1} = OB_t(1+i) - K_{t+1}$

principal repaid $PR_{t+1} = K_{t+1} - I_{t+1}$

$I_{t+1} = OB_t \times i.$

$PR_{t+1} = OB_t - OB_{t+1}$

3.1.9.

After first ten years, we still should repay $1000 = OB_0$.

payments: $1.5I_1, 1.5I_2, \cdots, 1.5I_{10}, \overbrace{X, X, \cdots, X}^{10}$

$OB_{10} = L$, $OB_{11} = OB_{10}(1+i) - 1.5I_1 = OB_{10}(1+i) - 1.5 OB_{10} \cdot i = OB_{10}(1-0.5i) = L(1-0.5i).$

$OB_{12} = OB_{11}(1+i) - 1.5I_2 = OB_{11}(1+i) - 1.5 OB_{11} \cdot i = OB_{11}(1-0.5i) = L(1-0.5i)^2.$

$OB_{20} = L(1-0.5i)^{10}.$

After the next ten years, we need to pay $OB_{20} = L(1-0.5i)^{10} = 1000(1-0.05)^{10} = 598.74$

$X a_{\overline{10}|0.1} = 598.74, \Rightarrow X = 97.44.$

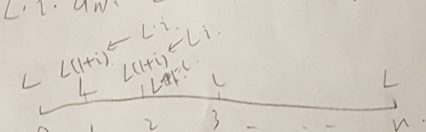
3.1.10.

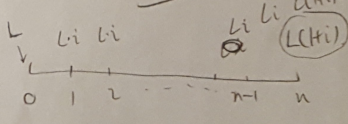
If everytime we only pay for interest, then the all of the payments are $L \cdot i$.

$L \cdot i \cdot a_{\overline{n}|i} + L \cdot v^n = L \cdot i \cdot \frac{1-v^n}{i} + Lv^n = L.$

$\cancel{L = L \cdot i \cdot a_{\overline{n-1}|i} + L(1+i)v^n}$

$L(1+i) = L + L i$.





$L \cdot v^n$

$\cancel{+ Li \cdot a_{\overline{n}|i}}$

3.2-1.

$L = K \cdot a_{\overline{n}|i}$,

$L(1+i)^t = K \cdot a_{\overline{n}|i}(1+i)^t = K \cdot \frac{1-v^n}{i} \cdot (1+i)^t = K \cdot \frac{(1+i)^t - v^{n-t}}{i} = K\left[\frac{(1+i)^t-1}{i} + \frac{1-v^{n-t}}{i}\right]$

$= K[s_{\overline{t}|i} + a_{\overline{n-t}|i}] = K s_{\overline{t}|i} + K a_{\overline{n-t}|i}$ $(L(1+i)^t - K s_{\overline{t}|i} = K a_{\overline{n-t}|i}.)$.

By definition, $OB_t = L(1+i)^t - K s_{\overline{t}|i} = K a_{\overline{n-t}|i}$.

$L(1+i)^t - K s_{\overline{t}|i} = L(s_{\overline{t}|i} \cdot i + 1) - K s_{\overline{t}|i} = (Li - K) s_{\overline{t}|i} + L = L + (I_1 - K) s_{\overline{t}|i}$

$= L - PR_1 s_{\overline{t}|i}$

$\left(\frac{(1+i)^t-1}{i} = s_{\overline{t}|i} \Rightarrow (1+i)^t = s_{\overline{t}|i} \cdot i + 1\right)$

3.2.2.

Quarterly payment is $\frac{3000}{a_{\overline{12}|0.02}} = 283.68$. Total interest paid $283.68 \times 12 - 3000 = 404.15$.

The original method has larger payments in the beginning, which reduces more outstanding balance so that the interest is also decreasing. So they don't need to pay that much interest.

3.2-3.

$OB_0 - OB_1 = PR_1 \Rightarrow OB_0 = PR_1 + OB_1 = 706 + 156 = 862$.

$I_1 = OB_0 \cdot i \Rightarrow i = \frac{I_1}{OB_0} = \frac{43.10}{862} = 0.05$, $I_2 = OB_1 \cdot i = 706 \times 0.05 = 35.30$.

payment $K = PR_1 + I_1 = 156 + 43.1 = 199.1$. $OB_2 = OB_1(1+i) - K = 592.20$. $PR_2 = K - I_2 = 163.80$.

$I_3 = OB_2 \cdot i = 27.11$ $PR_3 = K - I_3 = 199.1 - 27.11 = 171.99$. $OB_3 = OB_2(1+i) - K = 370.21$.

$I_4 = OB_3 \cdot i = 18.51$. $PR_4 = K - I_4 = 180.59$. $OB_4 = OB_3(1+i) - K = 189.62$.

$I_5 = OB_4 \cdot i = 9.48$, $PR_5 = K - I_5 = 189.62$. $OB_5 = OB_4(1+i) - K = 0$.

3.2-4.

Assume monthly payment is 1.

$L = OB_0 = 1 \cdot a_{\overline{60}|1\%} = a_{\overline{60}|1\%}$, $OB_t = a_{\overline{60-t}|1\%}$. by Table 3.6.

$a_{\overline{60-t}|1\%} < \frac{a_{\overline{60}|1\%}}{2} \Rightarrow t = 34.4$, we take $t = 35$. (June 1, 2007).

Tutorial: 3.2.6, 3.2.9, 3.2.12 Problem Set: 3.2.7, 3.2.8, 3.2.10, 3.2.11, 3.2.16.

3.2.7.

$X(1.06)^{10} - X$ is the total interest paid at the end of 10^{th} year.

$\frac{X}{a_{\overline{10}|6\%}}$ is the payment for each level

$$(X(1.06)^{10} - X) - \left(10 \cdot \frac{X}{a_{\overline{10}|6\%}} - X\right) = 356.54 \Rightarrow X = 825.$$

3.2.8.

(i) $K = \frac{2000}{a_{\overline{10}|8.07\%}} = 299 \Rightarrow 10k = 2990$

(ii) $200 + 2000i,\ 200 + 1800i, \ldots, 200 + 200i,$

total $2000 + 200i(10 + 9 + \cdots + 1) = 2000 + 200i \cdot \frac{10 \times 11}{2} = 2990 \Rightarrow i = 9\%.$

3.2.10.

(a)

Present value of interest: $\overset{I_1}{10v} + \overset{I_2}{8.99v^2} + \cdots + \overset{I_6}{7.29v^6} = 39.33$

Present value of principal: $\overset{PR_1}{105.61v} + \overset{PR_2}{106.67v^2} + \cdots + \overset{PR_6}{228.9v^6} = 560.67$

interest PV 3.4: $5(Da)_{\overline{7}|2} = 356.16.$

principal PV 3.4: $250\, a_{\overline{12}|} = 2643.84$

(b)

payment for each level is $K = \frac{L}{a_{\overline{n}|}}$, $I_t = k(1 - v^{n-t+1})$, $PR_t = kv^{n-t+1}$

PV of the interest: $\sum_{t=1}^{n} k(1 - v^{n-t+1}) \cdot v^t = L\left[1 - \frac{nv^{n+1}}{a_{\overline{n}|}}\right].$

PV of the principal: $\sum_{t=1}^{n} kv^{n-t+1} \cdot v^t = L \cdot \frac{nv^{n+1}}{a_{\overline{n}|}}$

3.2.11.

$OB_t = 5190.72,\ OB_{t+1} = 5084.68,\ OB_{t+2} = 4973.66.$

$PR_{t+1} = OB_t - OB_{t+1} = 106.04,\ PR_{t+2} = OB_{t+1} - OB_{t+2} = 111.02,$

$\Rightarrow 1 + j = \frac{PR_{t+2}}{PR_{t+1}} = 1.047. \quad I_{t+1} = OB_t \times i = 5190.72 \times 4.7\% = 243.77$

$K = I_{t+1} + PR_{t+1} = 349.81.$

3.2.16.

(a) (i) $K = \frac{L}{a_{\overline{n}|i}} = \frac{Li}{1-v_i^n}$. annual payment

(ii) $J = \frac{L}{a_{\overline{12n}|j}} = \frac{Lj}{1-v_j^{12n}}$, monthly payment

since $(1+j)^{12} = 1+i \Rightarrow v_j^{12n} = v_i^n$,

for (i) $OB_t = K \cdot a_{\overline{n-t}|i} = L \cdot \frac{a_{\overline{n-t}|i}}{a_{\overline{n}|i}} = L \cdot \frac{1-v_i^{n-t}}{1-v_i^n}$

for (ii) $OB_t = J \cdot a_{\overline{12(n-t)}|j} = L \frac{1-v_j^{12(n-t)}}{1-v_j^{12n}} = L \frac{1-v_i^{n-t}}{1-v_i^n}$ same as (i).

(b)

for (i) $I_T^{(i)} = \Sigma I_t = \Sigma (OB_{t-1} \times i) = i\,\Sigma (OB_0 + OB_1 + \cdots + OB_{n-1})$

for (ii) $I_T^{(ii)} = j\,\Sigma (OB_0' + OB_1' + \cdots + OB_{12n-1}')$.

since $j = \frac{i^{(12)}}{12} \le \frac{i}{12} \Rightarrow i \geq 12j$.

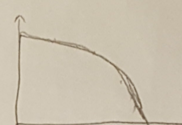
$I_T^{(i)} = 12j\,\Sigma (OB_0 + OB_1 + \cdots + OB_{n-1})$

Since $O + OB_1' + \cdots + OB_{11}' + OB_{12}' > 12\,OB_1$

$(OB_1' + \cdots + OB_{12}')j > 12\,OB_1 j$.

— — — — — ·

$I_T^{(ii)}$



$I_T^{(i)} = \sum_{t=1}^{n} I_t = \sum_{t=1}^{n} K(1-v_i^{n-t+1}) = \sum_{t=1}^{n} \frac{Li}{1-v_i^n}(1-v_i^{n-t+1})$.

$I_T^{(ii)} = \sum_{t=1}^{12n} I_t = \sum_{t=1}^{12n} J(1-v_j^{12n-t+1}) = \sum_{t=1}^{12n} \frac{Lj}{1-v_j^{12n}}(1-v_j^{12n-t+1})$

Tutorial 3.3.6s, 4.1.4, 4.1.12, 4.1.14 Problem set 3.3.1, 3.3.2, 3.3.3, 3.3.4, 3.4.1, 3.4.2, 4.1.1, 4.1.2, 4.1.3, 4.1.5, 4.1.6, 4.1.7, 4.1.8, 4.1.9, 4.1.10, 4.1.13.

3.3.1.

the total periodic outlay is $L(i + \frac{1}{s_{\overline{n}|j}})$

$i = 12\%,\ n = 10,\ j = 8\%.$

$$(X - 12{,}000)\, s_{\overline{5}|8\%} \cdot (1+8\%)^5 + (2X - 12{,}000)\, s_{\overline{5}|8\%} = 100{,}000 \Rightarrow X = 13{,}454.36$$

3.3.2.

(a) $16{,}902.95 = L(10\% + \frac{1}{s_{\overline{10}|8\%}}) \Rightarrow L = 100{,}000.$

(b) $K = L(i + \frac{1}{s_{\overline{n}|j}}) \Rightarrow L = \frac{K\, s_{\overline{n}|j}}{1 + i \cdot s_{\overline{n}|j}}$

3.3.3.

$10{,}000 = 100\, s_{\overline{n}|0.09/12} \Rightarrow n = 74.9,\quad 10{,}000 = 100\, \ddot{s}_{\overline{74}|0.0075} + X \Rightarrow X = 81.67$

X is the payment in last year

Total paid is $10{,}000(\frac{0.15}{12}) \times 75 + \underbrace{100 \times 74 + 81.67} = 16{,}856.67.$

3.3.4.

(i) $\frac{250{,}000}{a_{\overline{10}|0.12}} = 31{,}035.91$

(ii) $250{,}000(0.10 + \frac{1}{s_{\overline{10}|j}}) \Rightarrow j = 0.21322.$

$= 31{,}035.91$

3.4.1

$$PV = 10{,}000(v^{12} + v^{24} + \cdots + v^{96}) + 800(v + v^2 \cdots + v^{12}) + 700(v^{13} + \cdots + v^{24}) + 600(v^{25} + \cdots + v^{36})$$
$$+ \cdots + 100(v^{85} + \cdots + v^{96}).\quad i = 1\%.\quad v = \frac{1}{1+i}.$$

3.4.2

Makeham's formula $A = K + \frac{i}{j}(L - K)$

(a) $K = 1000(v_{3\%}^{4} + v_{3\%}^{8} + \cdots v_{3\%}^{60}) = 6615.21.$

$PV = K + \frac{i}{j}(L - K) = 6615.21 + \frac{4\%}{3\%}(15{,}000 - 6615.21) = 17.795.$

(b) $K = 1000(v^4 + 2v^8 + \cdots + 5v^{20}) = 9837.49 \Rightarrow PV = 16.723.$

(c) $K = 1000(3v^4 + \cdots + v^{20}) = 11{,}509.22 \Rightarrow PV = 16.165$

4.1.1.

$P = C \times \frac{1}{(1+j)^n} + Fr\left(\frac{1}{1+j} + \frac{1}{(1+j)^2} + \cdots + \frac{1}{(1+j)^n}\right) = Cv_j^n + Fr a_{\overline{n}|j}$.

$C = F$, $\underline{P = Fv_j^n + Fr a_{\overline{n}|j}}$, $v_j^n = 1 - j a_{\overline{n}|j}$, $P = C + (Fr - Cj) a_{\overline{n}|j}$.

$C = F$ in this case. $P = F + F(r-j) a_{\overline{n}|j}$. j : 6-month yield rate, r : coupon rate

(a), (b), (c), (d) have same F, $j_{(a)} < j_{(b)}$, $\Rightarrow a_{\overline{n}|j(a)} > a_{\overline{n}|j(b)}$. $\Rightarrow \underset{\hat{0}}{(r_{(a)} - j_{(a)})} a_{\overline{n}|j(a)} < (r_{(b)} - j_{(b)}) a_{\overline{n}|j(b)}$

then $P_{(a)} < P_{(b)}$. Similarly, $P_{(c)} < P_{(d)}$.

4.1.2.

$P = 115.84$, $F = 100$, $r = \frac{7\%}{2}$, $j = \frac{6\%}{2}$, $n = 24$.

$P = Cv_j^n + Fr a_{\overline{n}|j} = C v_{3\%}^{24} + 100 \times 3.5\% \, a_{\overline{24}|3\%} \Rightarrow C = 115$.

4.1.3.

$5083.49 = 10{,}000 \, v_j^{20} \Rightarrow j = 0.0344$.

$X = 10{,}000 \, v_j^{20} + 10{,}000 \times \frac{10\%}{2} a_{\overline{20}|j} = 12{,}229$.

4.1.5.

coupon payment : $1000 \times \frac{8\%}{2} = 40$,

accumulated value of reinvested coupon $40 \, s_{\overline{20}|6\%/2} = 1074.81$

After 10 yr, Don receives another redemption 1000, total is 2074.81.

then $900(1+j)^{20} = 2074.81 \Rightarrow j = 0.0426$, annual yield rate is 0.085.

4.1.6.

$P = 800$, $F = 1000$, $r = \frac{10\%}{4}$, $n = 25 \times 4 = 100$

$P = Fv_j^{100} + Fr a_{\overline{100}|j} \Rightarrow 800 = 1000 \, v_j^{100} + 25 a_{\overline{100}|j} \Rightarrow j = 0.0316$

$4 \times 0.0316 = 0.1264$

4.1.7.

$F = 1000$, $r = \frac{10\%}{2} = 5\%$, $n = 10 \times 2 = 20$. yield rate $j = \frac{8\%}{2} = 4\%$.

$P = 1000 \cdot v_{0.04}^{20} + 50 \, a_{\overline{20}|0.04} = 1135.90$.

Loan repayment $1135.90(1+7\%)^{10} = 2234.49$

reinvested coupons $50 \, s_{\overline{20}|6\%/2} = 1343.52$, plus redemption $1343.52 + 1000$ minus 2234.49 $= 109.43$

as net gain.

4.1.8.

$r = \frac{4.75\%}{2}$, $j = \frac{4.855\%}{2}$, $F = 100$. $n = 5 \times 2 = 10$.

$P = 100 v_j^{10} + 100 r\, a_{\overline{10}|j} = 99.539$.

4.1.9.

(a) $j = \frac{3.69\%}{2}$, $Fr = \frac{4.25}{2} = 2.125$.

$100 v_j^{5.5 \times 2} + 2.125\, a_{\overline{11}|j} = 102.76$.

(b) if $j = \frac{3.655\%}{2}$, $P = 102.79$, if $j = \frac{3.695\%}{2}$, $P = 102.74$.

(c) if $P = 102.765$, $j = 3.69\%$, if $P = 102.755$, $j = 3.692\%$.

4.1.10.

$P = F + F(r - j)\, a_{\overline{n}|j}$.

$95.59 = 100 + 100(0.02375 - j)\, a_{\overline{2n}|j}$.

$108.82 = 100 + 100(0.03125 - j)\, a_{\overline{2n}|j}$

thus, $j = 0.02625$, $j^{(2)} = 0.0525$.

4.1.13.

By $P = F + F(r - j)\, a_{\overline{n}|j}$.

$79.30 = 100 + 100(7\%/2 - j)\, a_{\overline{2n}|j}$, ... ①

$93.10 = 100 + 100(9\%/2 - j)\, a_{\overline{2n}|j}$ $\Rightarrow j = 0.05$

plug back in ①, then $n = 24$. or 12 years.

Additional Problem Set. 4.2.4, 4.2.5, 4.2.7 Tutorial: 5.1.1, 5.1.3, 5.1.4, 5.1.7

4.2.4.

yield rate $j = \frac{i^{(2)}}{2} = 3.3\%$, $BV_t = 90$, $F = 100$, $r = \frac{5\%}{2} = 2.5\%$,

then $BV_{t+1} = BV_t(1+j) - Fr = 90(1+3.3\%) - 100 \times 2.5\% = 90.47$.

4.2.5.

The entry under Principal Repaid is called the amount for amortization of premium. The amortization of a bond bought at a premium is also referred to as writing down a bond.

The premium $F(r-j)\, a_{\overline{n}|j} = 36$. $j = \frac{7\%}{2} = 3.5\%$.

the present value: $F(r-j)v_j^n = 1 \cdot v_j^4 \Rightarrow F(r-j)v_j^n = 0.8714$

$\Rightarrow n = 26$. 13 years.

$PR = K - I$

$= F \cdot r - F \cdot j$

4.2.7.

$PR_2 = 977.19$, $PR_4 = 1046.79$,

$PR_2(1+j)^2 = PR_4 \Rightarrow j = 0.035$ per half year.

$PR_1 = \frac{PR_2}{1+j} = 944.14$, then the premium is

$PR = PR_1\, s_{\overline{30}|0.035} = 48,739$.

Problem set 9: 5.1.5, 5.2.1, 5.2.2, 5.2.4, 5.2.5, Tutorial 5.2.3, 5.2.6, 5.31

5.1.5.

(a) $30{,}000 = \frac{14{,}000}{1+i} + \frac{12{,}000}{(1+i)^2} + \frac{6{,}000}{(1+i)^3} + \frac{4{,}000}{(1+i)^4} + \frac{2{,}000}{(1+i)^5} \Rightarrow i = 0.1203$

(b) MIRR: $L(1+j)^5 = KS$ 10%.

$3000(1+j)^5 = 14{,}000(1+10\%)^4 + 12{,}000(1+10\%)^3 + \cdots + 2{,}000 \Rightarrow j = 0.1081.$

(c) $NPV = \frac{14{,}000}{1.1} + \frac{12{,}000}{(1.1)^2} + \cdots + \frac{2{,}000}{(1.1)^5} - 30{,}000 = 1126$

(d) $14{,}000 + 12{,}000 + 6{,}000 > 30{,}000$ and $14{,}000 + 12{,}000 < 30{,}000.$

during 3rd year.

(e) $\frac{14{,}000}{1.1} + \cdots + \frac{2000}{(1.1)^5} > 30000, \quad \frac{14{,}000}{1.1} + \cdots + \frac{4000}{(1.1)^4} < 30{,}000.$

during 5th year.

(f) $I = \frac{\text{present value of cash inflows}}{\text{present value of outflows}}$

$$I = \frac{\frac{14{,}000}{1.1} + \cdots + \frac{2{,}000}{(1.1)^5}}{30{,}000} = 1.0375.$$

5.2.1.

The time-weighted return rate is $\left[\frac{F_1}{A} \times \frac{F_2}{F_1+C_1} \times \cdots \times \frac{F_k}{F_{k-1}+C_{k-1}} \times \frac{B}{F_k+C_k}\right] - 1,$

$$\sqrt{\frac{1{,}310{,}000 - 250{,}000}{1{,}000{,}000} \times \frac{1{,}265{,}000 + 150{,}000}{1{,}310{,}000} \times \frac{1{,}540{,}000 - 250{,}000}{1{,}265{,}000} \times \frac{1{,}420{,}000 + 150{,}000}{1{,}540{,}000}} -$$

$= 0.0910.$

5.2.2.

$\frac{12}{10} \cdot \frac{X}{12+X} - 1 = 0 \Rightarrow X = 60.$

$10(1+Y) + X(1+\frac{1}{2}Y) = X \Rightarrow Y = -0.25.$

5.2-4.

Jan 1st	50	
March 15	40	20
June 1	80	80
June 30	157.50.	

6-month time-weighted return is

$$\frac{40}{50} \times \frac{80}{40+20} \times \frac{157.50}{80+80} = 1.05$$

then ~~annual~~ annual return is $\sqrt{1.05} - 1 =$

One-year time weighted yield is

$$\frac{40}{50} \times \frac{80}{60} \times \frac{175}{160} \times \frac{X}{250} = 1.1025$$

$\Rightarrow X = 236.25.$

5.2-5.

Jan 1st	1
July 1st	0.8
Jan 1st	1.0

$\frac{0.8}{1} \times \frac{1.0}{0.8} - 1 = 0.$ time-weighted return is 0.

$$100{,}000(1+i) + 100{,}000(1+\tfrac{1}{2}i) = \underset{\text{July. 1st}}{100{,}000} \times \frac{1}{0.8} + \underset{\text{Jan 1st}}{100{,}000} \times 1$$

$= 225{,}000$

$\Rightarrow i = 0.1667.$

Problem Set 10: 6.1.1, 6.1.2, 6.1.3, 6.3.1, 6.3.2, 6.3.3, 6.3.4, 6.3.5, 6.3.6, 6.4.1, 6.4.2, 6.4.3.
Tutorial: 6.1.4, 6.1.5, 6.4.4.

6.1.1

$t_i = i$, $\quad C_1 (1+S_0(t_1))^{-t_1} + C_2 (1+S_0(t_2))^{-t_2} + \cdots + C_n (1+S_0(t_n))^{-t_n} = P.$

$$\frac{100 \times 10\%}{(1+S_0(1))^1} + \frac{100 \times 10\%}{(1+S_0(2))^2} + \frac{100 \times (1+10\%)}{(1+S_0(3))^3} = \frac{10}{1.15} + \frac{10}{1.1^2} + \frac{110}{(1.05)^3} = 111.98$$

Since coupon is paid once per year, the yield rate j satisfies.

$111.98 = 100 v_j^3 + 10 a_{\overline{3}|j} \Rightarrow j = 0.0556.$

6.1.2.

$S_0(1) = 0.1$, $S_0(2) = 0.1$, $S_0(3) = 0.12$, $S_0(4) = 0.12$, $F = 100$, $r = 5\%$.

$$P = \frac{Fr}{1+S_0(1)} + \frac{Fr}{(1+S_0(2))^2} + \frac{Fr}{(1+S_0(3))^3} + \frac{F(1+r)}{(1+S_0(4))} = \frac{5}{1.1} + \frac{5}{1.1^2} + \frac{5}{1.12^3} + \frac{105}{1.12^4} = 78.97.$$

6.1.3.

(a) (i) $\frac{Fr}{2} = \frac{100 \times 10\%}{2} = 5.$

$$\frac{5}{1+\frac{S_0(0.5)}{2}} + \frac{5}{(1+\frac{S_0(1)}{2})^2} + \cdots + \frac{105}{(1+\frac{S_0(3)}{2})^6} = 5\left(\frac{1}{1.0375} + \cdots + \frac{1}{1.0915^5}\right) + \frac{105}{1.0915^6} = 104.05$$

(ii) $\frac{5}{1.07} + \frac{5}{1.0675^2} + \cdots + \frac{105}{1.0675^6} = 93.15.$

(iii) $5 \cdot a_{\overline{6}|0.06} + 100(1.06)^{-6} = 95.08.$

(b) (i) $5 a_{\overline{6}|j} + 100 v_j^6 = 104.05 \Rightarrow j^{(2)} = 2j = 8.44\%.$

(ii) $5 a_{\overline{6}|j} + 100 v_j^6 = 93.15 \Rightarrow j^{(2)} = 2j = 12.82\%.$

(iii) $j = S_0(0.5) = 12\%$

6.3.1.

Forward rate of interest: $1 + i_0(n-1, n) = \frac{(1+S_0(n))^n}{(1+S_0(n-1))^{n-1}}.$

(a) $i_0(k-1, k) = \frac{(1+S_0(k))^k}{(1+S_0(k-1))^{k-1}} - 1.$

(b) $1 + i_0(0,1) = \frac{(1+S_0(1))^1}{(1+S_0(0))^0}$, $1 + i_0(1,2) = \frac{(1+S_0(2))^2}{(1+S_0(1))^1}$, $\ldots$, $1 + i_0(k-1,k) = \frac{(1+S_0(k))^k}{(1+S_0(k-1))^{k-1}}.$

$(1 + i_0(0,1)) \cdots (1 + i_0(k-1,k)) = (1+S_0(k))^k.$

(c) $\frac{d}{dS_0(k)} i_0(k-1,k) = \frac{d}{dS_0(k)} \cdot \frac{(1+S_0(k))^k}{(1+S_0(k-1))^{k-1}} = \frac{k(1+S_0(k))^{k-1}}{(1+S_0(k-1))^{k-1}} > 0$, and

$\frac{d}{dS_0(k-1)} i_0(k-1,k) = \frac{d}{dS_0(k-1)} \frac{(1+S_0(k))^k}{(1+S_0(k-1))^{k-1}} = \frac{-(k-1)(1+S_0(k))^k}{(1+S_0(k-1))^k} < 0.$

(d) If $S_0(k) > S_0(k-1)$, then $i_0(k-1,k) = (1+S_0(k))\left(\frac{1+S_0(k)}{1+S_0(k-1)}\right)^{k-1} - 1 > (1+S_0(k)) - 1$
$= S_0(k)$

6.3.3.

$F = 100$. for 6 months $S_0(1) = \frac{100}{97.8} - 1 = 0.02249$.

for 1-year $S_0(2) = \frac{100}{95.4} - 1 = 0.04822$.

$2 \times \left(\frac{(1+S_0(2))^2}{1+S_0(1)} - 1\right) = 0.149$. 6-month forward rate.
$\wedge\, i(1,2)$.

6.3.4.

$i(1,2) = \frac{(1+0.10)^2}{1+0.08} - 1 = 0.1204$. $\quad i(2,3) = \frac{(1+0.11)^3}{(1+0.10)^2} - 1 = 0.1303$

6.3.5.

(a) (i) $i_0(1,2) = \frac{(1+7\%)^2}{1+6\%} - 1 = 0.0801$

(ii) $i_0(2,3) = \frac{(1+9\%)^3}{(1+7\%)^2} - 1 = 0.1311$

(b) $i_0(3,4) = \frac{(1+S_0(4))^4}{1.09^3} - 1 \geq 0.1311 \Rightarrow S_0(4) \geq 0.1001$.

6.3.6.

$P = 5\left(\frac{1}{1.1} + \frac{1}{1.1^2} + \frac{1}{1.12^3} + \frac{1}{1.12^4}\right) + 105 \cdot \frac{1}{(1+S_0(5))^5}$, since $P = 73.68$, then

$S_0(5) = 0.125$. $\quad i_0(4,5) = \frac{(1+0.125)^5}{(1+0.12)^4} - 1 = 0.1452$.

4.1.

From (i) we receive $\frac{1000}{1+10\%} = 909.09$. after (ii) $909.09(1.08)^2 = 1060.36$.

909.09 1000 1060.36
0 1 2

$\frac{1060.36}{1000} - 1 = 6.04\%$
(a).

4.2.

$S_0(2) = 7\%$, $S_0(1) = 6\%$, since $i(1,2) = \frac{(1+S_0(2))^2}{1+S_0(1)} - 1 = 0.08$. ① Sell one-year zero coupon bond get $\frac{1}{1+6\%} = 0.9434$.
② Invest 0.9434 to two-year bond $0.9434 \times (1.07)^2 = 1.0801$, ③ borrow 1 with "someone" to pay the sold coupon bond.
④ pay the loan 1.07, get $1.08 - 1.07 = 0.01$ without any initial investment.

4.3.

(i) $\frac{1000}{(1.10)^2} = 826.45$ (ii) $826.45 \times (1+8\%) = 892.56$ $\quad i(1,2) = \frac{1000}{892.56} - 1 = 12\%$. (b).

7.2.2.

$PV_L(0.10) = 300,000$, $n=5$. $X \cdot a_{\overline{5}|0.1} = 300,000 \Rightarrow X = 79,139$. annual payments

$n=15$, $X \cdot a_{\overline{15}|0.1} = 300,000 \Rightarrow X = 39,442$

$n=50$, $X \cdot a_{\overline{50}|0.1} = 300,000 \Rightarrow X = 30,257$.

$n=100$, $X \cdot a_{\overline{100}|0.1} = 300,000 \Rightarrow X = 30,002$.

7.2.7.

(a) Asset = Liability, present value $= \frac{A_1}{1.1} + \frac{A_5}{(1.1)^5} = 100\left(\frac{1}{1.1^2} + \frac{1}{1.1^4} + \frac{1}{1.1^6}\right) = 207.39$.

By $PV_A(i_0) = \sum_0^n A_t v_{i_0}^t = \sum_0^n L_t v_{i_0}^t = PV_L(i_0)$, $\sum_0^n (A_t - L_t)(1+i_0)^{n-t} = 0$.

$\frac{d}{di} PV_A(i)\Big|_{i_0} = \frac{d}{di} PV_L(i)\Big|_{i_0}$. then $-\frac{A_1}{(1.1)^2} - \frac{5A_5}{(1.1)^6} = 100\left(-\frac{2}{1.1^3} - \frac{4}{1.1^5} - \frac{6}{1.1^7}\right)$

$\Rightarrow \frac{A_1}{1.1} + \frac{5A_5}{1.1^5} = 777.18$

$\Rightarrow A_1 = 71.44$, $A_5 = 229.41$.

(b) The condition is $\frac{d^2}{di^2} PV_A(i)\Big|_{i_0} > \frac{d^2}{di^2} PV_L(i)\Big|_{i_0}$

$\frac{2A_1}{1.1} + \frac{6\times 5A_1}{(1.1)^5} = 4403 \quad > \quad 100\left(\frac{2\times 1}{1.1^2} + \frac{5\times 4}{1.1^4} + \frac{6\times 5}{1.1^6}\right) = 3225$

9.1.1.

(a) No-arbitrage forward price $S_0 e^{rT} = 2000 \times e^{5\% \times 1} = 2102.54$.

(b) ① Take the short position on one-year forward contract in price 2150.

② Borrow 2000 to buy one ounce

③ After one year, give the platinum to buyer, and pay back $2000e^{0.05} = 2102.54$ to lender, get profit $2150 - 2102.54 = 47.46$.

(c) Since no arbitrage, portfolio A: long position + $Ke^{-r(T-t)}$ cash Portfolio B. 1 unit asset.

After $T-t$, value A is $K + f_{T,t}$, value B is $K + f_{T,t}$, then present value

$f_t = S_t - Ke^{-r(T-t)} = 2000 - 2102.54e^{-0.05\times 0.5} = -50.63$.

9.1.2.

(a) $P_1 = S_0 e^{rT} = 900e^{0.08} = 974.96$, $P_2 = 900e^{0.08\times 2} = 1056.16$.

(b) At time 1, 2-year long forward contract $= S_1 - 900e^{1\times(0.08)}$

3-year short contract $900e^{(3-2)(0.08)} - S_1$ combine is 0.

(c) 2-year long $S_1 - 900e^{2(0.08)}e^{-0.10}$ 3-year short $900e^{3(0.08)}e^{-2(0.10)} - S_1$

Combined value is -18.92.